

# A NOTE ON A NEW EXPONENTIAL BOUND FOR M-ACCEPTABLE RANDOM VARIABLES

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**ABSTRACT. English :** We present a new exponential inequality as a generalization of that of Sung *et al.* [14] for  $M$ -acceptable random variables, and hence for extended negative ones. Our result is based on the simple real inequality  $e^x \leq 1 + x + (|x|/2)e^{|x|}, x \in \mathbb{R}$ , in place of the following one:  $e^x \leq 1 + x + (x^2/2)e^{|x|}, x \in \mathbb{R}$ , used by many authors. We compare the given bound with former ones.

**French :** Nous généralisons la notion d'acceptabilité d'une variable aléatoire à celle de  $M$ -acceptabilité, puis établissons une généralisation de l'inégalité exponentielle de Sung *et al.* [14] pour ce type de variables, contenant aussi les variables étendument négatives. Notre résultat se base sur l'inégalité simple suivante  $e^x \leq 1 + x + (|x|/2)e^{|x|}, x \in \mathbb{R}$ , à la place de celle-ci :  $e^x \leq 1 + x + (x^2/2)e^{|x|}, x \in \mathbb{R}$ , utilisée par plusieurs auteurs. Nous comparons notre borne exponentielle avec d'autres précédemment introduites.

## 1. INTRODUCTION

Let  $\{X_n, n \geq 1\}$  be a sequence of random variables defined on a fixed probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . An exponential inequality for the partial sum  $\sum_{i=1}^n (X_i - \mathbb{E}X_i)$  was introduced and proved by Sung *et al.* [14] for a class of dependent random variables, that is for  $M$ -acceptable random variables. This type of inequality provides a measure of the convergence rate for the strong law of large numbers. It is rather classical for the independent case but seems very recent in the dependent case.

We refer to Sung *et al.* [13] for an extended introduction on dependent random variables. We will only recall basic definitions to make this paper self-contained. First, recall that a finite family of random variables  $\{X_i, 1 \leq i \leq n\}$  is positiveley associated if for any real functions  $f_1$  and  $f_2$  defined on  $\mathbb{R}^n$  and coordinatewise increasing, we have

$$\text{Cov}(f_1(X_1, \dots, X_n), f_2(X_1, \dots, X_n)) \geq 0.$$

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This type of dependence was introduced in Esary, Proschan and Walkup [3]. Next the negative association was set up by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [5]. We have negative association whenever for any pair of disjoint subsets  $A$  and  $B$  of  $\{1, 2, \dots, n\}$ , with respective cardinalities  $p$  and  $q$ , and for any pair functions respectively defined of  $\mathbb{R}^p$  and  $\mathbb{R}^q$ , we have

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0.$$

Further, the concept of extended negatively dependent random variables was proposed by Liu [8] as a generalization of negative dependence introduced by Lehmann [7], meaning that the family  $\{X_1, \dots, X_n\}$  is extended negatively dependent if there is some  $M > 0$  such that for any real numbers  $x_1, \dots, x_n$ , we have :

$$\mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n) \leq M \prod_{i=1}^n \mathbb{P}(X_i \leq x_i)$$

and

$$\mathbb{P}(X_1 > x_1, \dots, X_n > x_n) \leq M \prod_{i=1}^n \mathbb{P}(X_i > x_i).$$

The special case  $M = 1$  corresponds to the Lehman original definition. Finally, Sung *et al.* [14] defined another negative dependence denamed *acceptable* for a family  $\{X_1, \dots, X_n\}$  for which there exists  $\delta > 0$  such that for any real  $\lambda$  such that  $|\lambda| \leq \delta$ ,

$$(1.1) \quad \mathbb{E} \exp\left(\lambda \sum_{i=1}^n X_i\right) \leq \prod_{i=1}^n \mathbb{E} \exp(\lambda X_i).$$

Before we go further, we recall that from the steps of Joag - Dev and Proschan [5], that a number of well known multivariate distributions fulfils the negative association property, such as the coordinates of a multinomial vector, that of a multivariate hypergeometric. Also the following stochastic processes satisfy the negative dependence : Dirichlet; permutation distribution; negatively correlated normal distributions; random sampling without replacement and joint distribution of ranks; etc.

Now as to the probability inequality field, as pointed out by Sung *et al.* [14], exponential inequalities for positively associated random variables were obtained by Devroye [2], Ioannides and Roussas [4], Oliveira [10], Sung [12]), Xing and Yang (2008) ; Xing, Yang and Liu [18]. We may also cite the following other contributors in this field: Kim and Kim

[6], Nooghabi and Azarnoosh [9], Roussas [11], Sung [13], Xing [15], Xing and Yang [17], Xing *et al.* [19]. Many of these inequalities are generalized by Sung *et al.* [14], particularly for negatively associated random variables.

In this note we aim at improving significantly the current exponential inequalities into two directions. The first is to extend the acceptability of Sung *et al.* [14] by *M-acceptability*, for  $M > 0$ , by replacing (1.1) by

$$(1.2) \quad \mathbb{E} \exp\left(\lambda \sum_{i=1}^n X_i\right) \leq M \prod_{i=1}^n \mathbb{E} \exp(\lambda X_i).$$

It is clear that extending these inequalities to *M-acceptable* random variable may result in a significant generalization as we point out below. The second generalization concerns the inequality itself. We will see in the next section that our results are based on a much precise real analysis inequality. For example Sung *et al.* [14] used the inequality

$$(1.3) \quad \forall (x \in R), e^x \leq 1 + x + \frac{x^2}{2} e^{|x|}.$$

To obtain a better exponential inequality for negative random variables, we will use this inequality

$$(1.4) \quad \forall (x \in R), e^x \leq 1 + x + \frac{|x|}{2} e^{|x|}.$$

The remainder of this paper is organized as follows. In Section 2, we state our results and make some comments on their generality. The proofs, including that of (1.4), are given in Section 3. It is worth mentioning that most of the methods of Sung *et al.* [14] are reconducted here and hence, will not detailed as it should be.

## 2. RESULTS AND COMMENTS

**2.1. Results.** As pointed out in the introduction, the inequality (1.4) is essential. It is used to prove the

**Lemma 1.** *Let  $X$  be a random variable with  $\mathbb{E} e^{\delta|X|} < \infty$  for some  $\delta > 0$ . Then for any  $0 < \lambda \leq \delta/2$ ,*

$$\mathbb{E} e^{\lambda(X - \mathbb{E}X)} \leq \exp(K\lambda)$$

*where  $K$  is defined as  $K = (\mathbb{E}|X|^2)^{1/2} \mathbb{E} e^{\delta|X|}$ .*

This lemma is itself a generalization of Lemma 1 in Sung *et al.* [14] (2011). We will get the new exponential inequality from it later.

**Theorem 1.** *Let  $\{X_n, n \geq 1\}$  be a sequence of identically distributed  $M$  acceptable random variable with  $\mathbb{E} e^{\delta|X_1|} < \infty$  for some  $\delta > 0$ .*

*Then for any  $\varepsilon \geq K$ ,*

$$\mathbb{P}\left(\left|\sum_{i=1}^n (X_i - \mathbb{E} X_i)\right| > n \varepsilon\right) \leq 2 M \exp(-n \delta/2(\varepsilon - K)),$$

where  $K = (\mathbb{E}|X_1|^2)^{1/2} \mathbb{E} e^{\delta|X_1|}$ .

## 2.2. Comments.

**2.2.1. Comparison with Sung et al. results.** For the special case of negatively associated random variables, let us compare our result with those of Sung et al. [14]. In their Remark 2.2, Sung et al. [14] pointed out that Sung [13] upper bound and their own one, are of the same convergence rate when  $\varepsilon$  is bounded. Our results results based on an unbounded  $\varepsilon$  is hence more general than those in the two papers. Next, for  $\varepsilon_n = 2 (K\alpha \log n/n)^{1/2}$ ,  $\alpha > 0$  and  $K = 2(E|X_1|^2)^{1/2} E e^{\delta|X_1|}$ , they obtained the bound  $2\exp(-\alpha \log n)$  in their Theorem 2.2. They remarked its supiority to those of Kim and Kim [6], Nooghabi and Azarnook [9], Xing [15], Xing et al. [19] and Xing and Yang [17]. It is clear that our upper bound is better than that of Sung et al. [14] and then to all the cited ones above.

**2.2.2. Particular case.** By choosing  $\varepsilon = n^{\alpha-1} + K$  in Theorem 1, we have the following result

**Theorem 2.** *Let  $\{X_n, n \geq 1\}$  be a sequence of identically distributed  $M$  acceptable random variables with  $\mathbb{E} e^{\delta|X_1|} < \infty$  for some  $\delta > 0$ . Let  $\varepsilon_n = n^{\alpha-1} + K$ , where  $\alpha > 0$  and  $K = (\mathbb{E}|X_1|^2)^{1/2} \mathbb{E} e^{\delta|X_1|}$ . Then for all  $n$ ,*

$$\begin{aligned} \mathbb{P}\left(\left|\sum_{i=1}^n (X_i - \mathbb{E} X_i)\right| > n \varepsilon_n\right) \\ \leq 2 M \exp\left(-\frac{\delta}{2} n^\alpha\right) \end{aligned}$$

**Proof.** Let  $\varepsilon_n = n^{\alpha-1} + K$ , where  $\alpha > 0$  and  $K = (E|X_1|^2)^{1/2} E e^{\delta|X_1|}$ . Then  $\varepsilon_n - K \geq 0$  for all  $n$ . Hence the result follows from Theorem 1.

## 3. PROOFS

Let us begin with the proof of Inequality (1.4).

**3.1. Proof of (1.4).** We first remark that for all  $x \in R$ ,

$$(3.1) \quad \frac{|x|}{2} e^{|x|} = \frac{|x|}{2} + \frac{|x|^2}{2} + \sum_{n=2}^{\infty} \frac{|x|^{n+1}}{2n!}.$$

Next, we remark that for  $n \geq 3$ ,  $2(n-1)! \leq n(n-1)! = n!$ , and then  $1/n! \leq 1/(2(n-1)!)$  and next

$$\frac{x^n}{n!} \leq \frac{|x|^n}{n!} \leq \frac{|x|^n}{2(n-1)!}$$

and finally

$$(3.2) \quad \sum_{n=3}^{\infty} \frac{x^n}{n!} \leq \sum_{n=3}^{\infty} \frac{|x|^n}{2(n-1)!} = \sum_{n=3}^{\infty} \frac{|x|^n}{2(n-1)!}.$$

To conclude, we use the decomposition

$$e^x = 1 + x + \frac{|x|^2}{2} + \sum_{n=3}^{\infty} \frac{|x|^n}{2(n-1)!}$$

and use (3.1) and (3.2) to get

$$\begin{aligned} e^x &= 1 + x + \frac{|x|^2}{2} + \sum_{n=3}^{\infty} \frac{x^n}{n!} \leq e^x \leq 1 + x + \frac{|x|^2}{2} + \sum_{n=3}^{\infty} \frac{|x|^n}{2(n-1)!} \\ &= 1 + x + \frac{|x|}{2} (|x| + \sum_{n=3}^{\infty} \frac{|x|^{n-1}}{(n-1)!}) = 1 + x + \frac{|x|}{2} (|x| + \sum_{n=2}^{\infty} \frac{|x|^n}{n!}) \\ &= 1 + x + \frac{|x|}{2} (\sum_{n=1}^{\infty} \frac{|x|^n}{n!}) \leq 1 + x + \frac{|x|}{2} (\sum_{n=0}^{\infty} \frac{|x|^n}{n!}) = 1 + x + \frac{|x|}{2} e^{|x|}. \end{aligned}$$

Now let us turn to the proof of Lemma 1.

**3.2. Proof of Lemma 1.** In view of inequality (1.4) and by using the Hölder inequality for  $0 < \lambda \leq \delta/2$ , to get

$$\begin{aligned} \mathbb{E}e^{\lambda(X-\mathbb{E}X)} &\leq 1 + \lambda\mathbb{E}(X-\mathbb{E}X) + \frac{\lambda}{2}\mathbb{E}(|X-\mathbb{E}X|) e^{\lambda|X-\mathbb{E}X|} \\ &= 1 + \frac{\lambda}{2}\mathbb{E}(|X-\mathbb{E}X|) e^{\lambda|X-\mathbb{E}X|} \\ &\leq 1 + \frac{\lambda}{2}(\mathbb{E}(X-\mathbb{E}X)^2)^{1/2}(\mathbb{E}e^{2\lambda|X-\mathbb{E}X|})^{1/2}. \end{aligned}$$

Now

$$(\mathbb{E}e^{2\lambda|X-\mathbb{E}X|})^{1/2} \leq (\mathbb{E}e^{2\lambda|X|+2\lambda|\mathbb{E}X|})^{1/2} \leq (\mathbb{E}e^{2\lambda|X|})^{1/2}(e^{2\lambda|\mathbb{E}X|})^{1/2}$$

and by Jensen inequality,  $e^{2\lambda|\mathbb{E}X|} \leq \mathbb{E}e^{2\lambda|X|}$ , we end up with

$$(\mathbb{E}e^{2\lambda|X-\mathbb{E}X|})^{1/2} \leq \mathbb{E}e^{2\lambda|X|}.$$

Next,

$$\mathbb{E}(X - \mathbb{E}X)^2)^{1/2} \leq (\mathbb{E}X^2 - (\mathbb{E}X)^2)^{1/2} \leq (\mathbb{E}X^2)^{1/2}.$$

Putting all this together, we get

$$\mathbb{E}e^{\lambda(X-\mathbb{E}X)} \leq 1 + \frac{\lambda}{2}(\mathbb{E}X^2)^{1/2}\mathbb{E}e^{2\lambda|X|} \leq 1 + \frac{\lambda}{2}K \leq \exp(\frac{\lambda}{2}K).$$

This achieves the proof. Let us close the proofs section by that of Theorem 1.

**3.3. Proof of Theorem 1.** Let  $\varepsilon \geq K$ . By Markov's inequality, by the definition of  $M$  acceptable random variables and by the Lemma, we have for  $0 < \lambda \leq \delta/2$ ,

$$\begin{aligned} \mathbb{P}(\sum_{i=1}^n (X_i - \mathbb{E} X_i) > n \varepsilon) &= \mathbb{P}(\exp(\lambda \sum_{i=1}^n (X_i - \mathbb{E} X_i)) > \exp(\lambda n \varepsilon)) \\ &\leq \exp(-\lambda n \varepsilon) \mathbb{E} \exp(\lambda \sum_{i=1}^n (X_i - \mathbb{E} X_i)) \\ &\leq M \exp(-\lambda n \varepsilon) \prod_{i=1}^n \mathbb{E}(\exp(\lambda(X_i - \mathbb{E} X_i))) \leq M \exp(-\lambda n \varepsilon) \prod_{i=1}^n \exp(K\lambda), \end{aligned}$$

since the  $X_i$  have the same first and second moments. We finally get

$$\mathbb{P}(\sum_{i=1}^n (X_i - \mathbb{E} X_i) > n \varepsilon) \leq M \exp(-\lambda n \varepsilon + \lambda K n).$$

The right member is minimum for  $\lambda = \delta/2$  since  $\varepsilon \geq K$ . We then get for this value

$$\mathbb{P}(\sum_{i=1}^n (X_i - \mathbb{E} X_i) > n \varepsilon) \leq M \exp(-n \delta/2 (\varepsilon - K)).$$

Since  $\{-X_n, n \geq 1\}$  are also  $M$  acceptable random variables, we can replace  $X_i$  by  $-X_i$  in the above statement that is ,

$$(3.3) \quad \mathbb{P}(-\sum_{i=1}^n (X_i - \mathbb{E} X_i) > n \varepsilon) \leq M \exp(-n \delta/2 (\varepsilon - K)).$$

Now observing that  $|x| = \max(x, -x)$  and  $(|x| > a) = (x > a \text{ or } x < -a)$ , we arrive at

$$\begin{aligned} & \mathbb{P}\left(\left|\sum_{i=1}^n (X_i - \mathbb{E} X_1)\right| > n \varepsilon\right) \\ & \leq \mathbb{P}\left(\sum_{i=1}^n (X_i - \mathbb{E} X_n) > n \varepsilon\right) + \mathbb{P}\left(-\sum_{i=1}^n (X_i - \mathbb{E} X_n) > n \varepsilon\right) \\ & \leq 2M \exp(-n \delta/2 (\varepsilon - K)). \end{aligned}$$

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